Julia set describes quantum tunnelling in the presence of chaos

2002 J. Phys. A: Math. Gen. 35 L225

(http://iopscience.iop.org/0305-4470/35/17/101)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) L225–L231 PII: S0305-4470(02)34589-X

LETTER TO THE EDITOR

Julia set describes quantum tunnelling in the presence of chaos

A Shudo[1], **Y Ishii**[2] **and K S Ikeda**[3]

[1] Department of Physics, Tokyo Metropolitan University, Minami-Ohsawa, Hachioji, Tokyo 192-0397, Japan
[2] Department of Mathematics, Kyushu University, Hakozaki, Fukuoka 812-8581, Japan
[3] Faculty of Science and Engineering, Ritsumeikan University, 1916 Noji-cho, Kusatsu 525-0055, Japan

Received 4 March 2002
Published 19 April 2002
Online at stacks.iop.org/JPhysA/35/L225

Abstract

We find that characteristics of quantum tunnelling in the presence of chaos can be regarded as a manifestation of the Julia set of the complex dynamical system. Several numerical pieces of evidence for the standard map, together with a rigorous statement for the Hénon map, are presented, demonstrating that the complex classical paths which contribute to the semiclassical propagator are dense in the Julia set. Chaotic tunnelling can thus be characterized by the transitivity of the dynamics and high density of the trajectories on the Julia set.

PACS numbers: 03.65.−w, 05.45.−a

Recent studies on tunnelling in multidimensions have revealed that the existence of chaos affects the signature of quantum tunnelling. The observation of purely quantum mechanical calculation in chaotic systems shows that tunnelling can become *chaotic* or chaos seems to *assist* tunnelling [1–11]. The idea capturing such a novel aspect of tunnelling appears very attractive, but a direct connection between chaos and tunnelling can only be accomplished by interpreting the quantum phenomenon by the trajectory of the classical dynamics [6–8, 10, 11]. When one is particularly interested in the tunnelling process, the use of complex trajectories is essential since the transition due to tunnelling occurs where the real-valued classical trajectories cannot reach.

The best known technique using the complex space is the so-called instanton method in which the tunnelling penetration is evaluated mainly by a single classical path moving on the reversed potential [12, 13]. On the other hand, in chaotic systems, it has been found in the time-domain semiclassical analysis that a bunch of complex paths contribute almost equally to the tunnelling transition between classically forbidden regions [10, 11]. They typically form a treelike fractal structure in the complex initial-value plane, and its outstanding appearance compels us to prepare some concept which controls dominating complex paths in the semiclassical sum of contributing candidates [10, 11]. All the characteristic structures appearing in the tunnelling wavefunction in chaotic systems originate from it.

However, the *Laputa chain*, which was so introduced in [10, 11], is still phenomenological so far, and even remains mysterious if no link to some concept compatible with the dynamical system theory is made. One may thus naturally ask why such a structure plays a special role in the complex trajectory description of chaotic tunnelling, and what sort of mechanism underlies such conspicuous objects in the complex plane. The purpose of this letter is to provide a clear answer to these questions. Our final claim is simple and would be rather natural: *the Julia set is the origin of chaotic tunnelling*.

Let us begin by introducing the model system we are concerned with. The system we study here is a family of two-dimensional area preserving maps, in which the mixed phase space is realized in a certain range of the parameter space. The time evolution of the phase point (p, θ) is given as the mapping rule as

$$(p_{n+1}, \theta_{n+1}) = F(p_n, \theta_n) \equiv (H'(p_n) - V'(\theta_n), \theta_n + H'(p_n) - V'(\theta_n)). \qquad (1)$$

Here, $H(p) = p^2/2$ and $V(\theta) = K \sin\theta$ are the most standard choice, but suitable modification or replacement of the kinetic or the potential term is sometimes helpful and will be made according to the target of the analysis.

Since the map model does not have the energy as the Hamiltonian flow problem does, one cannot consider the tunnelling through the energetic barrier, which may be a normal setting of the tunnelling problem. Instead, dynamical confinement due to classically disconnected components such as KAM tori and chaotic components in the phase space plays the role of barriers, and the quantum transition between such invariant regions is regarded as tunnelling [14].

At least in the first setting of the problem, it is not at all obvious that several different situations, such as the tunnelling transition out of the quasiperiodic region into some chaotic component, or its reverse process, or that between different chaotic components, could be treated on the same footing. However, as will be described below and as will also become one of the most important points in this letter, the choice of initial and final states does not matter to the whole story.

Typical quantum mechanical wavefunctions in the mixed phase space are displayed in figure 1. In both models, the tails of the wavefunctions do not monotonically decay even in the tunnelling regime; rather there appear several unexpected structures: the crossovers of the slope, the plateau regions and irregular interference patterns on it. All these characteristics are only qualitatively featured [10, 11], but they are commonly observed not only in the dynamical tunnelling problem, but also in the energetic barrier tunnelling [15].

The semiclassical approach, which is extensively developed in recent studies of quantum chaos or quantum chaology [16], works quite well even when one employs it as a tool describing the tunnelling process. Apart from an added technical (but sometimes crucial) problem originating from the Stokes phenomenon [17], which we do not enter into details of here, our task in the semiclassical analysis is essentially the same as the real one, that is, to evaluate the Van-Vleck propagator:

$$\Psi_n(p_0, p_n) \approx \sum_{\substack{p_0=\alpha \\ p_n=\beta}} A_n(p_0, \theta_0) \exp\left\{-\frac{\mathrm{i}}{\hbar} S_n(p_0, \theta_0)\right\}, \qquad (2)$$

where the summation is taken for all (p_0, θ_0) which satisfy the boundary conditions for the initial momentum $p_0 = \alpha$ and the final momentum $p_n = \beta$. Here, $S_n(p_0, \theta_0) = \sum_{j=1}^{n}$

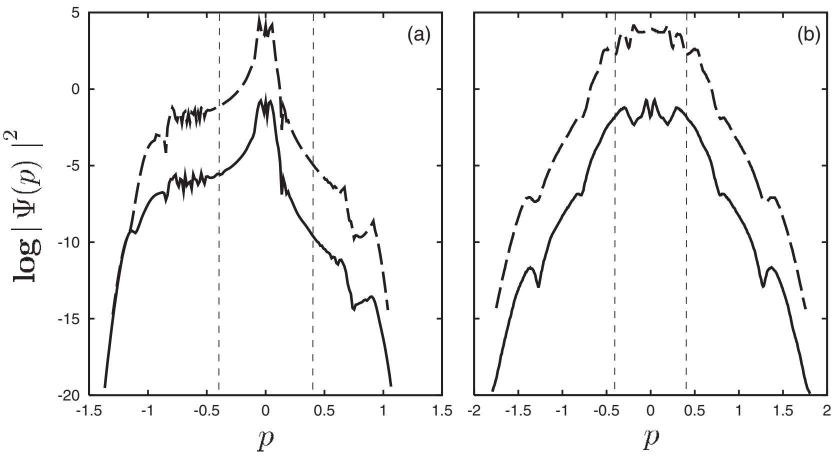

Figure 1. Quantum (full curve) and semiclassical (dashed curve) wavefunction for (*a*) the model with $H_0(p) = (p^2/2)(p/p_d)^6/[(p/p_d)^6 + 1] + \omega p$ and $V(\theta) = K \sin\theta$, where $p_d = 5$, $\omega = 2$ and $K = 1.2$, and (*b*) the model with $H_0(p) = p^2/2$ and $V(\theta) = K \sin\theta$, where $K = 1.5$. In both cases, the initial wavepacket is set as $\Psi(p) = \delta(p)$. The real-valued classical orbits cannot reach the region outside the dashed lines within the time step taken here ($n = 6$ for (*a*) and 5 for (*b*)). The semiclassical wavefunction is shifted in order to clarify the structure.

$[H(\theta_j) - V(\theta_j) + \theta_j(p_j - p_{j+1})]$ is the action along a classical trajectory, and $A_n(p_0, \theta_0) = [2\pi\hbar(\partial p_n/\partial\theta_0)_{p_0}]^{-\frac{1}{2}}$ represents the amplitude factor associated with its stability.

Since we here take the p-representation, p_0 should be a real quantity. So, the canonical partner θ_0 may be used to identify the (complexified) trajectories contributing to the sum (2), and it is then allowed to be complex as $\theta_0 = \xi + \mathrm{i}\eta$ (ξ, η real). We visualize the contributing complex paths by displaying the set [10, 11]:

$$\mathcal{M}_n \equiv \bigcup_{\beta\in\mathbb{R}} \mathcal{M}_n^{*,\beta} = \bigcup_{\beta\in\mathbb{R}} \{(p, \theta) \in \mathbb{C}^2 \mid p_n = \beta\} \tag{3}$$

on the θ_0-plane of the slice $\{p_0 = \alpha\}$ for some initial condition $\alpha \in \mathbb{R}$. The set $\mathcal{M}_n$ on the θ_0-plane, which usually looks like clouds or a wisteria trellis on a macroscopic scale, is decomposed into finer and finer structures as it is magnified [10, 11]. One can see that its basic element is a string with various scales. Each string represents a trajectory appearing in the semiclassical sum (2). We note that in the integrable limit only the branches connected with the real plane (i.e. $\eta = 0$) survive and all other complicated objects disappear [10, 11].

A huge number of candidate paths may discourage us since it appears to be no longer possible to establish a simple view of tunnelling in the presence of chaos. However, among all possible candidates the complex paths forming a sequential structure, which runs in the vertical direction at the centre of figure 2(*a*) and is clearly discernible from the other aggregated strings, exceed any other candidate paths in amplitude. We have called such a characteristic structure the *Laputa chain* [10, 11]. As shown in figure 1, one finds that the semiclassical sum including *only* such complex paths as contained in the Laputa chains has reproduced almost all details of tunnelling into chaotic regions. Our task is, therefore, reduced to clarifying what this marked structure appearing in the initial-value plane represents.

The reason why some complex paths dominate the others in the semiclassical sum (2) is, in general, that the imaginary parts of their action, Im $S_n(p_0, \theta_0)$, are relatively small. This is because the absolute value of each term in (2) is mainly governed by Im $S_n(p_0, \theta_0)$, rather than the amplitude factor $A_n(p_0, \theta_0)$. This in turn means that the complex paths forming the

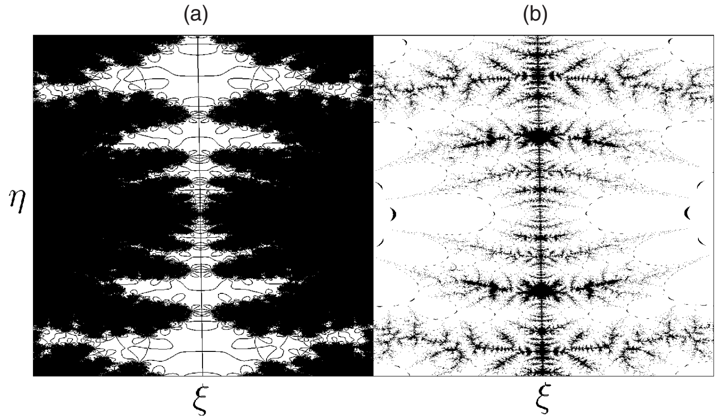

Figure 2. (*a*) A magnification of the initial-value representation $\mathcal{M}_n$ on the θ_0-plane in the case of the standard map with $K = 1.5$ and $n = 40$. The range shown above is given as $4.197\,487\,346 \times 10^{-2} \leqslant \xi \leqslant 4.197\,487\,362 \times 10^{-2}$, and $6.693\,592 \times 10^{-4} \leqslant \eta \leqslant 6.693\,601 \times 10^{-4}$. The initial momentum is set as $p_0 = 0$. The $\mathcal{M}_n$-set consists of a bunch of self-similar objects, whose basic element is a string. Except for the central part, such strings are so densely aggregated that individual strings cannot be resolved in this scale. However, magnifying the black area, one can find a similar structure in the scale shown here, that is, the black area is also composed of a bunch of string objects. Each string represents an individual component of the semiclassical sum (2). The strings running in the vertical direction look as if they cross with each other, but actually they avoid with very narrow gaps; that is, the strings form a serial chainlike structure connected via narrow gaps. (*b*) The slice of K^+ by $\{p_0 = \alpha\}$. This is numerically obtained by plotting the initial points whose trajectories remain a ball in $\mathbb{C}^2$ with a certain finite radius, $r = 10^3$, in this case.

Laputa chain should gain small imaginary action as compared with the other paths not forming the chain structure. Indeed, as shown later, the trajectories initially placed on the Laputa chain approach the real (p, θ)-plane exponentially, which provides minimal or relatively small imaginary action.

Conversely, one can say that this property characterizes the Laputa chain and makes these paths distinguishable from the others. Furthermore, they are specific in that these trajectories stay in bounded regions because after coming close to the real plane they almost follow the behaviour of the trajectories on the real plane and the real orbits are all bounded in the present situation.

This is a hint to link the Laputa chain to a proper object compatible with the theory of dynamical systems, since the Julia set, which plays a central role in the complex dynamical systems, is specified as the set satisfying such a property. More precisely, the *forward Julia set* J^+ is defined as the boundary of the set K^+ of points whose forward orbits remain in a finite region [18]:

$$K^+ = \{\, (p, \theta) \mid \{F^n(p, \theta)\}_{n>0} \text{ is bounded } \} \tag{4}$$

and

$$J^+ = \partial K^+. \tag{5}$$

A polynomial diffeomorphism like the Hénon map f, which is defined on $\mathbb{C}^2$, has a polynomial inverse, so both the forward and the backward iterations can be considered. In such a case we define K^+ (K^-) as the set of points in $\mathbb{C}^2$ whose forward (backward) orbits are bounded, and J^+ (J^-) to be the boundary of K^+ (K^-), which we call the *forward* (*backward*) *Julia set*. The set $J \equiv J^+ \cap J^-$ is called the *Julia set* of f. The forward (or backward) Julia set $K^\pm$ is where the orbits have sensitive dependence on initial conditions, which means that the chaotic motion is realized on it.

Remarkably enough, such a purely mathematical object enters physics as quantum tunnelling in chaos. Indeed, as shown in figure 2(*b*), the similarity between the chain-shaped structure demonstrated in figure 2(*a*) and the slice of K^+ by the same plane is obvious. The slice of K^+ shows a typical dendrite-like structure, which often appears in one-dimensional complex dynamical systems [19]. The number of strings constituting the Laputa chain increases at an exponential rate, so coincidence between them becomes better as the time proceeds [11, 23]. Note that the orbits put on the highly aggregated branches surrounding the Laputa chain do not stay in a finite phase space domain but rapidly escape to infinity.

It is possible to provide a rigorous statement if one focuses on the cubic potential model given by putting $H(p) = p^2/2$ and $V(\theta) = c\theta - \theta^3/3$. The map (1) is transformed to a standard form of the Hénon map,

$$f : (x,\ y) \longmapsto (y,\ y^2 + (1-c) - x), \tag{6}$$

by the affine change of coordinate $(p, \theta) = (y - x,\ y - 1)$. The Hénon map is known to be one of the simplest nonlinear systems in two-dimensional space, and its dynamics is extensively studied by several authors. Among them, investigation from the complex dynamical point of view has been developed in the last decade (see, for example, [20–22] and references therein) by using the pluripotential theory, the theory of currents etc.

As in the case of the standard map, it is reasonable to focus on the Im $S_n(p_0, \theta_0)$ of each trajectory, but to be compatible with the invariant set of the dynamical system one should consider the set of trajectories having the property described above in the limit of n going to infinity. The most natural condition would be to select the complex orbits whose Im $S_n(p_0, \theta_0)$ has a finite limit even when n goes to infinity. Such a filtering only serves as a necessary condition for semiclassically contributing orbits, but it is at least true that the trajectories whose Im $S_n(p_0, \theta_0)$ are divergent cannot contribute to the semiclassical summation since these orbits either tend to zero in their magnitude or will be removed by the Stokes phenomenon [17]. Therefore we define the *Laputa chains* as

$$\mathcal{C}_{\mathrm{Laputa}} \equiv \{(p, \theta) \in \mathcal{M}_\infty \mid \mathrm{Im}\ S_n(p, \theta) \text{ converges absolutely at } (p, \theta)\}. \tag{7}$$

In this definition $\mathcal{M}_\infty$ is an object introduced to represent the limit of $\mathcal{M}_n$-set when n goes to infinity. More precisely,

$$\mathcal{M}_\infty \equiv \bigcup_{\beta \in \mathbb{R}} \mathcal{M}_\infty^\beta, \tag{8}$$

where $\mathcal{M}_\infty^\beta$ is given as the Hausdorff limit of $\mathcal{M}_n^{*,\beta} \equiv \left\{(p, \theta) \in \mathbb{C}^2 \mid p_n = \beta\right\}$ (compare equation (3)). Thus, the set $\mathcal{M}_\infty$ corresponds to $\mathcal{M}_n$ for the time step '$n = \infty$'. It is possible to prove that this Hausdorff limit itself contains the forward Julia set J^+ [11, 23], which in itself is a partial verification of our numerical observation. The following assertion concerning the relation between $\mathcal{C}_{\mathrm{Laputa}}$ thus defined and J^+ is proved by the second-named author.

Theorem. *Let F be the time-one map on $\mathbb{C}^2$ associated with the kicked rotor (1) with $H(p) = p^2/2$ and $V(\theta) = c\theta - \theta^3/3$, and $h_{top}(F)$ be the topological entropy with respect to F,*

(i) If $h_{\mathrm{top}}(F|_{\mathbb{R}^2}) > 0$, then $\overline{\mathcal{C}_{\mathrm{Laputa}}} \supset J^+$.
(ii) If F is hyperbolic on J and $h_{\mathrm{top}}(F|_{\mathbb{R}^2}) > 0$, then $\overline{\mathcal{C}_{\mathrm{Laputa}}} = J^+$.
(iii) If F is hyperbolic on J and $h_{\mathrm{top}}(F|_{\mathbb{R}^2}) = \log 2$, then $\mathcal{C}_{\mathrm{Laputa}} = J^+$.

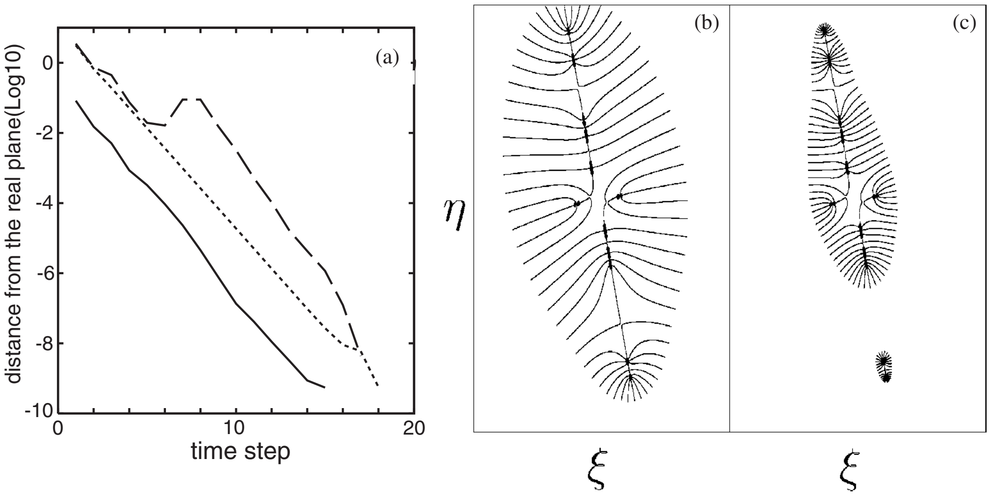

Figure 3. (*a*) The distance from the real plane as a function of time is displayed for the orbits whose initial conditions are put on the Laputa chains. The solid curve denote the case of the standard map. The dotted and broken curves are those for the Hénon map. When the Julia set J exists only on the real plane, the orbits always approach the real plane directly (dotted curve), but otherwise the orbits first move around in $\mathbb{C}^2$ space, and then approach the real plane (broken curve). (*b*), (*c*) The set $\bigcup_{\beta<\beta_0} \mathcal{M}_n^{*,\beta}$ ($\beta_0 = 10^{10}$) for the Hénon map is shown as the solid curves in the case of (*b*) $n = 9$ and (*c*) $n = 10$. The slice of J^+ by $\{p = \alpha\}$ is shown as the dots in each figure.

Here $\overline{X}$ indicates the closure of the set X. The rough sketch of the proof is as follows [11, 23]: that $h_{\rm top}(F|_{\mathbb{R}^2}) > 0$ implies the existence of a saddle periodic point X in the real phase space. A principal tool we shall employ is the following result, which was established by Bedford and Smillie [20–22]. For a complex one-dimensional locally closed submanifold M in either J^+ or an algebraic variety, there is a constant $c > 0$ so that

$$\lim_{n\to+\infty} \frac{1}{2^n}[f^{-n}M] = c \cdot dd^c G^+ \tag{9}$$

in the sense of current, where $[M]$ is the current of integration of M, i.e. $[M](\phi) \equiv \int_M \phi|_M$, and dd^c is the complex Laplacian. In this statement, G^+ represents the Green function for K^+ given by

$$G^+(x, y) \equiv \lim_{n\to+\infty} \frac{1}{2^n} \log^+ \left\| f^n(x, y) \right\| . \tag{10}$$

It is easily shown that the support of μ^+ coincides with J^+. From this theorem we see that the stable manifold of any periodic saddle p is dense in J^+, that is, $\overline{(W^s(p))} = J^+$. Using this result, together with the fact that the Hausdorff limit $\mathcal{M}_\infty$ contains J^+, we obtain the desired claim.

This claim gives a mathematical verification of the observation found numerically. Indeed, as shown in figure 3(*a*), the trajectories leaving the Laputa chains approach exponentially the real (p, θ)-plane. This makes $\mathrm{Im}\, S_n(p_0, \theta_0)$ converge absolutely. As a demonstration of the theorem, we show in figures 3(*b*) and (*c*) the set $\bigcup_{\beta<\beta_0} \mathcal{M}_n^{*,\beta}$ for a fixed β_0 and the slice of J^+ by $\{p_0 = \alpha\}$ for the Hénon map. One can see how the $\mathcal{M}_n$-set shrinks to the J^+ as a function of n.

Notice that the assumption $h_{\rm top}(F|_{\mathbb{R}^2}) > 0$ in the above theorem is a mathematical expression which corresponds to the fact that the underlying classical dynamics $F|_{\mathbb{R}^2}$ is chaotic.

We also note that the slice of the forward Julia set J^+ by $\{p = \alpha\}$ can be shown to have positive capacity for any initial condition $\alpha \in \mathbb{R}$. Thus, the theorem above suggests that, unlike the instanton solutions in the integrable case, a bunch of paths in $\mathbb{C}^2$ contributes to the tunnelling phenomena if the underlying classical mechanics is chaotic.

It should be noted that the assumption in (i) covers the system with mixed phase space, which is the most generic situation in physics. In addition, the physical implication or interpretation of another theorem of Bedford and Smillie [20–22] on the transitivity of the dynamics is suggestive in our problem. It states that for any $\mathbb{C}^2$-neighbourhoods of any two points in the chaotic regions there is an orbit in $\mathbb{C}^2$ connecting them, even in the case where the chaotic regions in the real plane are mutually disjointed by KAM tori. This property exactly guarantees the transition between any disconnected regions on the real-valued classical dynamics, and non-zero tunnelling amplitude of the wavefunction in arbitrary regions is always realized due to the transitivity on the Julia set.

In this way, with the help of strong mathematical claims, which could be established only by extending the dynamics to the complex space, we can clearly understand the reasons why chaos seems to assist tunnelling and can become chaotic; these can be attributed to such *high density* of the tunnelling paths in J^+ and the *transitivity* of the complexified dynamics. So far, the structure of the Julia set has been an object which mainly attracts the interest of mathematicians, but the present result implies that the Julia set is really observable as chaotic tunnelling in various physical and chemical phenomena.

References

[1] Bohigas O, Tomsovic S and Ullmo D 1990 *Phys. Rev. Lett.* **64** 1479
[2] Bohigas O, Tomsovic S and Ullmo D 1993 *Phys. Rep.* **223** 45
[3] Tomsovic S and Ullmo D 1994 *Phys. Rev.* E **50** 145
[4] Doron E and Frischat S D 1995 *Phys. Rev.* E **75** 3661
[5] Creagh S C and Whelan N D 1996 *Phys. Rev. Lett.* **77** 4975
[6] Creagh S C and Whelan N D 1999 *Phys. Rev. Lett.* **82** 5237
[7] Creagh S C 1998 *Tunneling in Complex Systems* ed S Tomsovic (Singapore: World Scientific) p 103
[8] Takahashi K and Ikeda K S 2001 *Found. Phys.* **31** 177
[9] Takahashi K and Ikeda K S 2000 *Ann. Phys., NY* **283** 94
[10] Shudo A and Ikeda K S 1995 *Phys. Rev. Lett.* **74** 682
[11] Shudo A and Ikeda K S 1998 *Physica* D **115** 234
[12] Callan C G Jr and Coleman S 1977 *Phys. Rev.* D **16** 1762
[13] Miller W H 1975 *J. Chem. Phys.* **62** 1899
[14] Davis M J and Heller E 1981 *J. Chem. Phys.* **75** 246
[15] Onishi T, Shudo A, Ikeda K S and Takahashi K 2001 *Phys. Rev.* E **64** 025201(R)
[16] Gutzwiller M C 1990 *Chaos in Classical and Quantum Mechanics* (New York: Springer)
[17] Shudo A and Ikeda K S 1996 *Phys. Rev. Lett.* **76** 4151
[18] Hubbard J H 1986 *Notes Rep. Math. Sci. Eng.* vol 2 (New York: Academic) p 101
[19] Devaney R L 1989 *An Introduction to Chaotic Dynamical Systems* 2nd edn (Reading, MA: Addison-Wesley)
[20] Bedford E and Smillie J 1991 *Invent. Math.* **103** 69
[21] Bedford E and Smillie J 1991 *J. Am. Math. Soc.* **4** 657
[22] Bedford E and Smillie J 1992 *Math. Ann.* **294** 395
[23] Shudo A, Ishii Y and Ikeda K S, in preparation